LETTER

A Phasor Model with Resting States

Teruyuki MIYAJIMA†, *Member*, **Fumihiko BAISHO**††, **Kazuo YAMANAKA**†, *Nonmembers*, **Kazuhiko NAKAMURA**††, *and* **Masahiro AGU**†††, *Members*

SUMMARY A new phasor model of neural networks is proposed in which the state of each neuron possibly takes the value at the origin as well as on the unit circle. A stability property of equilibria is studied in association with the energy landscape. It is shown that a simple condition guarantees an equilibrium to be asymptotically stable.

key words: *neural networks, associative memory, complex-valued neuron, stability analysis, Lyapunov theory*

1. Introduction

In some recently proposed neural network models the state of each neuron takes the value on the unit circle on the complex plane [1]–[6]. A complex-value on the unit circle can be considered as a phase of pulse sequences from a firing neuron. Thus, the network model with local state (state of a neuron) on the unit circle is called phasor model. On the other hand, neurons are allowed to have the resting state "0" as well as the firing state "1" in the binary networks proposed earlier, e.g. [7]. This observation makes us recognize the lack of the resting states in phasor models. A method for introducing the resting states is to enable neurons to take the value at the origin as well as on the unit circle. This leads to a new model with complex states, which we discuss in this paper. The model considered here will be referred to as *phasor model with resting states*. The phasor model with resting states has more capability for processing information, for example, smooth curves can be displayed on a two-dimensional array of phasors with resting state.

One major application of neural networks is associative memory in which an equilibrium corresponds to a memory. A symmetrically connected phasor model has equilibria of the global neural state as in the well known binary Hopfield model [7]. However, an equilibrium is useful as a memory only if it is stable. Agu et al.[5] studied a stability property of equilibria in terms of the energy landscape of the phasor model, and showed a relation between the neural connection and stability of an equilibrium. It is not straightforward to claim a similar result for the phasor model with resting states because this model has a quite different property from the phasor model, e.g. the energy for the global state change does not always decrease unlike the phasor model. In this letter, we introduce a particular notion of neighborhood that enables us to consider the local behavior of the network in a manner similar to the previous paper [5]. Furthermore, we obtain a stronger result on stability.

Manuscript received December 9, 1998.
Manuscript revised August 10, 1999.

†The authors are with the Department of Systems Engineering, Ibaraki University, Hitachi-shi, 316–8511 Japan.
††The authors are with the Graduate School of Science and Engineering, Ibaraki University, Hitachi-shi, 316–8511 Japan.
†††The author is with the Department of Media and Telecommunication Engineering, Ibaraki University, Hitachi-shi, 316–8511 Japan.

2. Updating Rule

Consider a neural network with N neurons. The local state (output) of the jth neuron is denoted by x_j. A local state is allowed to be either at the origin, $x_j = 0$, or on the unit circle, $x_j = \exp(i\phi_j)$, where ϕ_j denotes the argument of x_j.

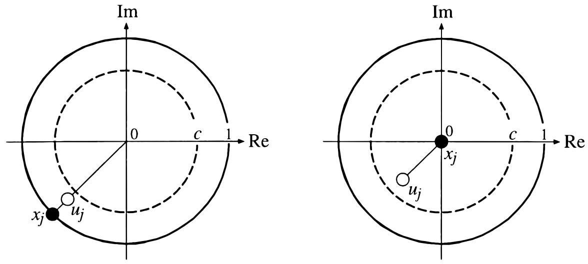
A local state is assumed to be asynchronously updated based only on its membrane potential. The membrane potential of the jth neuron is given by

$$u_j = \sum_{i=1}^{N} w_{ji} x_i \tag{1}$$

where w_{ji} represents the complex-valued connection weight between the jth and ith neuron. It is assumed that the hermitian property $w_{ji} = w_{ij}^*$ holds where $(\cdot)^*$ stands for the complex conjugate. $w_{jj} = 0$ is not assumed. Thus, the following results hold for $w_{jj} = 0$. When a neuron is updated the new local state is determined by the following rule:

$$x_j = \begin{cases} \exp(i\phi_j), \quad \phi_j \triangleq \arg(u_j) & (|u_j| \geq c) \\ 0 & (|u_j| < c) \end{cases} \tag{2}$$

where $c > 0$ is a given threshold. Figure 1 illustrates the relation between the destination of the state and the membrane potential of a neuron. To summarize the updating rule, we can state as follows: if the magnitude of the membrane potential is smaller than the threshold, let the local state be at the origin; otherwise, let the local state be on the unit circle with the same argument as the membrane potential.



(a) $|u_j| \geq c$ (b) $|u_j| < c$

Fig. 1 Updated state.

3. Stability Analysis

In order to discuss a stability property along Lyapunov's method, let us consider an energy function. The energy function is defined by

$$U(\mathbf{x}) \triangleq -\frac{1}{2}\sum_{j=1}^{N}\sum_{i=1}^{N} x_j^* w_{ji} x_i \tag{3}$$

where $\mathbf{x} \triangleq (x_1\ x_2\ \cdots\ x_N)$ denotes a global state. The energy decreases monotonically as the global state changes in the phasor model without resting states [5]. Let us consider the energy change for the global state changes in the phasor model with resting states. Suppose the jth neuron change its state. Denote the state before and after the transition by $\mathbf{x} = (x_1 \cdots x_j \cdots x_N)$ and $\mathbf{x}' = (x_1 \cdots x_j' \cdots x_N)$, respectively. Then, the change of the energy is given by

$$\Delta U = \mathrm{Re}(u_j^* x_j) - \mathrm{Re}(u_j^* x_j') - \frac{w_{jj}}{2}(|x_j|^2 - |x_j'|^2). \tag{4}$$

Thus the energy does not necessarily decrease due to the possible local-state transition between the unit circle and the origin. However, the energy function can play a similar role as long as local properties are considered as described below.

Consider an equilibrium $\bar{\mathbf{x}} = (\bar{x}_1\ \bar{x}_2\ \cdots\ \bar{x}_N) = (\exp(i\bar{\phi}_1)\ \exp(i\bar{\phi}_2)\ \cdots\ \exp(i\bar{\phi}_N))$. Let the set of indices of the local states on the unit circle be denoted by I, i.e.,

$$I \triangleq \{j|\ |\bar{x}_j| = 1\}$$

and the number of these neurons by $|I|$. We assume $0 < c < \min_{j\in I} |\bar{u}_j|$ where $\bar{u}_j$ is jth neuron's membrane potential at $\bar{\mathbf{x}}$. This assumption is needed for $\bar{\mathbf{x}}$ to be an equilibrium. Define δ-neighborhood of a state $\mathbf{x}_0 = (x_{01} \cdots x_{0N})$ as

$$\Omega \triangleq \{\mathbf{x}|\ x_j = 0,\ j \notin I \text{ and } |\arg(x_j) - \arg(x_{0j})| < \delta,\ j \in I\}$$

for arbitrary real number $\delta > 0$. That is, the δ-neighborhood of $\mathbf{x}_0$ is the set composed of the global states obtained from $\mathbf{x}_0$ by changing $\arg(x_{0j}), j \in I$ within $\pm\delta$ from $\bar{\phi}_j$ and leaving $x_{0j}, j \notin I$ at the origin. In the following we write "neighborhood Ω" if there is no need to specify δ. Considering the energy change in Ω, we now obtain the following result.

Lemma 1: The function U decreases so long as the state transition occurs in a neighborhood Ω of an equilibrium.

Proof: Suppose the jth component of $\mathbf{x} = (x_1 \cdots x_j \cdots x_N) \in \Omega$ changes to get to the new state $\mathbf{x}' = (x_1 \cdots x_j' \cdots x_N) \in \Omega$. Then, the change of the energy ΔU can be written as

$$\Delta U = \mathrm{Re}(u_j^* x_j) - \mathrm{Re}(u_j^* x_j') . \tag{5}$$

By assumption, if the jth local state x_j is on the unit circle, then x_j' is also on the unit circle. Since the argument of x_j' is the same as that of u_j from (2), the first term in (5) which is the inner product of u_j and x_j is always smaller than the second term. Hence, $\Delta U < 0$. □

In phasor models, since the global state $\bar{\mathbf{x}}\exp(i\theta)$ with an arbitrary θ is also an equilibrium, any one of the local states is clamped and the behavior of the other local states in a neighborhood of the equilibrium should be considered. As for the phasor model with N neurons, it has been shown in [5] that if the energy function is locally convex in a neighborhood of an equilibrium as a function of an arbitrary set of $N-1$ phase variables ϕ_j, the state stays in the neighborhood against small initial disturbance. As for the phasor model with resting states, neurons at the origin can be neglected since the neurons never affect to the dynamics so long as the state transition occurs in a neighborhood Ω of an equilibrium. In other words the phasor model with resting states can be regarded as a phasor model with $|I|$ neurons. Hence, we can expect to get a similar result on stability for the phasor model with resting states as in the phasor model. In fact, we get the following result.

Proposition 2: Choose an arbitrarily $i \in I$. Suppose that ϕ_i is fixed to $\bar{\phi}_i$ and U is convex in a neighborhood Ω as a function of the other $|I|-1$ variables $\phi_j, j \in I - \{i\}$. Let $\Omega_1 \subset \Omega$ be an arbitrary neighborhood of $\bar{\mathbf{x}}$. Then, all the sequences of the global states with initial states in a neighborhood $\Omega_0 \subset \Omega$ stay in Ω_1.

Proposition 2 ensures that the global state stays near an equilibrium $\bar{\mathbf{x}}$ if the initial state is sufficiently close to $\bar{\mathbf{x}}$. In order to prove Proposition 2, we have to take consideration into the state transition between the origin and the unit circle, unlike the phasor model. If such a transition never occurs, we can prove Proposition 2 along a similar way as in the phasor model [5]. We first show such a transition never occurs during one-step transition.

Now, let a sequence of the global states generated by (1) and (2) be denoted by $\mathbf{x}_0, \mathbf{x}_1, \cdots, \mathbf{x}_k, \cdots$, where the subscript k increases when the global state changes. This state transition can be described formally as follows:

$$\mathbf{x}_{k+1} = \Phi(\mathbf{x}_k). \tag{6}$$

Then, we get the following result.

Lemma 3: The function Φ is continuous at $\bar{\mathbf{x}}$ in terms of the neighborhood defined above.

Proof: Let $\delta' > 0$ be arbitrarily given and Ω' be δ'-neighborhood of $\bar{\mathbf{x}}$. Take a sufficiently small $\epsilon > 0$. Then, $|\Delta u_j| < \epsilon$ means

$$|\Delta u_j| < ||\bar{u}_j| - c|, \quad j = 1, \cdots, N$$

and

$$|\arg(\bar{u}_j + \Delta u_j) - \arg(\bar{x}_j)| < \delta', \quad j \in I.$$

Furthermore, take a sufficiently small $\delta > 0$, which may depend on ϵ. Then, $|\phi_j - \bar{\phi}_j| < \delta$ assures $|\Delta u_j| < \epsilon$, due to the continuous dependence of u_i on ϕ_j. This implies that if $\mathbf{x}$ is taken in δ-neighborhood Ω of $\bar{\mathbf{x}}$ then $\Phi(\mathbf{x}) \in \Omega'$. □

Proof of Proposition 2: Paying attention to the existence of the neurons at the origin, we can prove Proposition 2 by using Lemmas 1 and 3 along similar way as in [5]. □

In the above discussion, we have considered the stability of an equilibrium. Furthermore, we get the following result, which can be combined with Proposition 2 to mean asymptotic stability of the equilibrium.

Proposition 4: Choose an arbitrarily $i \in I$. Suppose that ϕ_i is fixed to $\bar{\phi}_i$ and U is convex in a neighborhood Ω as a function of the other $|I| - 1$ variables ϕ_j. Then, there exists a neighborhood $\Omega_0 \subset \Omega$ such that

$$\mathbf{x}_k \to \bar{\mathbf{x}} \quad (k \to \infty)$$

whenever the sequence of the global states starts from the inside of neighborhood Ω_0.

Proof: From Proposition 2, it is sufficient to consider only $|I|$ neurons. Then, we can use the energy as a Lyapunov function to derive the above result along a similar way to the well known stability theory, e.g. [8]. □

Hence, we can conclude that the equilibrium is asymptotically stable if the energy is locally convex.

Now a sufficient condition for the local convexity of the energy is derived as in the phasor model [5]. The elements of the Hessian matrix of the energy are given by

$$\frac{\partial^2 U}{\partial \phi_i^2} = \mathrm{Re}\left(x_i^* \sum_{j \neq i} w_{ij} x_j\right) \quad i, j \in I, \tag{7}$$

$$\frac{\partial^2 U}{\partial \phi_i \partial \phi_j} = -\mathrm{Re}\left(x_i^* w_{ij} x_j\right), \quad i \neq j, \quad i, j \in I \tag{8}$$

The sufficient condition is that all the $(|I|-1) \times (|I|-1)$ principal submatrices of the Hessian matrix are positive definite. A sufficient condition for the positive definiteness of the principal submatrices is

$$\mathrm{Re}(\bar{x}_j^* w_{ji} \bar{x}_i) > 0 \quad i, j \in I. \tag{9}$$

For example, this condition is satisfied when the connection weights are determined using the Hebbian rule to store a single pattern.

4. Conclusions

In this paper, we have considered three issues on phasor models. First, we have proposed a new phasor model where the state of each neuron possibly takes the value at the origin as well as on the unit circle. Second, it has been shown that analysis of local behavior of the new phasor model can be reduced to that of the phasor model by introducing a particular notion of neighborhood. Third, a stability result was obtained in a stronger sense of stability, i.e. asymptotically stability. Future study is left for the capacity of the proposed model as an associative memory network.

References

[1] A.J. Noest, "Associative memory in sparse phasor neural networks," Europhys. Lett., vol.6, no.6, pp.469–474, July 1988.

[2] A. Hirose, "Dynamics of fully complex-valued neural networks," Electronics Lett., vol.28, no.16, pp.1492–1494, July 1992.

[3] R.S. Zemel, C.K.I. Williams, and M.C. Mozer, "Lending direction to neural networks," Neural Networks, vol.8, no.4, pp.503–512, 1995.

[4] H. Aoki, "Analysis of equilibrium states of Hopfield associative memory extended to complex number field," IEICE Trans., vol.J78-A, no.9, pp.1238–1241, Sept. 1995.

[5] M. Agu, K. Yamanaka, and H. Takahashi, "A local property of the phasor model of neural networks," IEICE Trans. Inf. & Syst., vol.E79–D, no.8, pp.1209–1211, Aug. 1996.

[6] S. Jankowski, A. Lozowski, and J.M. Zurada, "Complex-valued multistate neural associative memory," IEEE Trans. Neural Networks, vol.7, no.6, pp.1491–1496, Nov. 1996.

[7] J.J. Hopfield, "Neural networks and physical systems with emergent collective computational abilities," Proc. Natl. Acad. Sci. USA, vol.79, pp.2554–2558, 1982.

[8] R.E. Kalman and J.E. Bertram, "Control system analysis and design via the "second method" of Lyapunov, II discrete-time systems," Trans. ASME, J. Basic Eng., vol.82D, no.2, pp.394–400, June 1960.